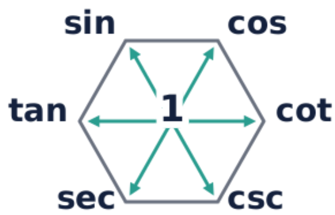
Name:______________________________ Date:____________________

Trigonometric Identities: Advanced

Conceptual Foundations 🧠

True mastery of identities requires understanding their origins, not just memorizing the final forms. Instead of simply filling in blanks, derive the connections between these fundamental ratios.



Derivation Challenge:

Start with the primary Pythagorean Identity: $\sin^2\theta + \cos^2\theta = 1$. Show the algebraic steps required to transform this equation into the secondary identity: $1 + \tan^2\theta = \sec^2\theta$.

Reverse Engineering & Domain Analysis

In this section, you will construct expressions to meet specific criteria and analyze the subtle domain restrictions that often get lost during simplification.

Constraint Challenge:

Construct a trigonometric expression that simplifies to **1**, subject to the constraint that you must use **at least three different** trigonometric functions (e.g., sin, sec, cot) in your original expression.

Critical Analysis:

Consider the simplification: **(sec x ÷ csc x) = tan x**.
Algebraically, the result is valid. However, the domains of the left side and right side are not identical. Identify the values of *x* in the interval [0, 2π] where the original expression is undefined but the simplified result (tan x) is defined.

Extending Formulas

🧠 **Optimization Strategy:** The cosine double angle formula has three variations. When proving identities, choosing the variation that eliminates a constant term (like '1') usually leads to the most efficient proof.

Apply compound angle formulas to derive a 'Triple Angle' identity.

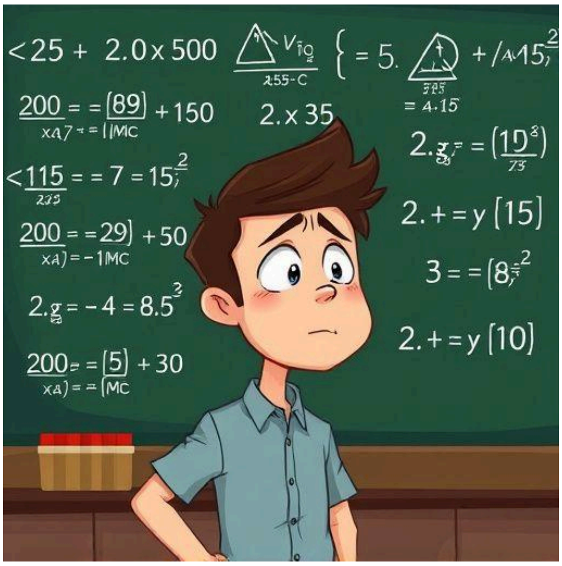
Prove that $\cos(3x) = 4\cos^3 x - 3\cos x$

Hint: Express 3x as (2x + x) and use the sum of angles formula first.

Rigorous Proof & Logic

Error Analysis:

A student claims that **$\sqrt{\sin^2 x + \cos^2 x} = \sin x + \cos x$**. Is this student correct? Explain the algebraic flaw in their reasoning using a numerical counter-example or algebraic rule.



Synthesis Proof

Prove the following identity. This requires confident manipulation of double angle formulas to group and factor terms.

$(1 - \cos 2x + \sin 2x) \div (1 + \cos 2x + \sin 2x) = \tan x$

Answer Key

Conceptual Foundations 🧠

Divide every term in the primary identity by $\cos^2\theta$:
$(\sin^2\theta/\cos^2\theta) + (\cos^2\theta/\cos^2\theta) = (1/\cos^2\theta)$
$\tan^2\theta + 1 = \sec^2\theta$

Reverse Engineering & Domain Analysis

Answers will vary. Example: (sin x * sec x) / tan x = 1
(sin x * 1/cos x) / (sin x/cos x) = tan x / tan x = 1

The original expression contains csc x (undefined at 0, π, 2π) and sec x (undefined at π/2, 3π/2). The result tan x is undefined at π/2, 3π/2.

Therefore, the 'holes' occur at x = 0, π, and 2π, where csc x is undefined but tan x (0) would be valid.

Extending Formulas

Answer:
$\cos(2x+x) = \cos 2x \cos x - \sin 2x \sin x$
$= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x$
$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$
$= 2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x$
$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$
$= 4\cos^3 x - 3\cos x$

Rigorous Proof & Logic

Incorrect. The student assumes that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, which is false.
LHS = $\sqrt{1}$ = 1.
RHS = sin x + cos x (which varies).
At x=0, LHS=1, RHS=1 (coincidentally true).
At x=π, LHS=1, RHS=-1 (false).

Numerator: $(1 - (1-2\sin^2 x) + 2\sin x \cos x) = 2\sin^2 x + 2\sin x \cos x = 2\sin x(\sin x + \cos x)$
Denominator: $(1 + (2\cos^2 x-1) + 2\sin x \cos x) = 2\cos^2 x + 2\sin x \cos x = 2\cos x(\cos x + \sin x)$
Result: $(2\sin x(\sin x+\cos x)) / (2\cos x(\cos x+\sin x)) = \sin x/\cos x = \tan x$